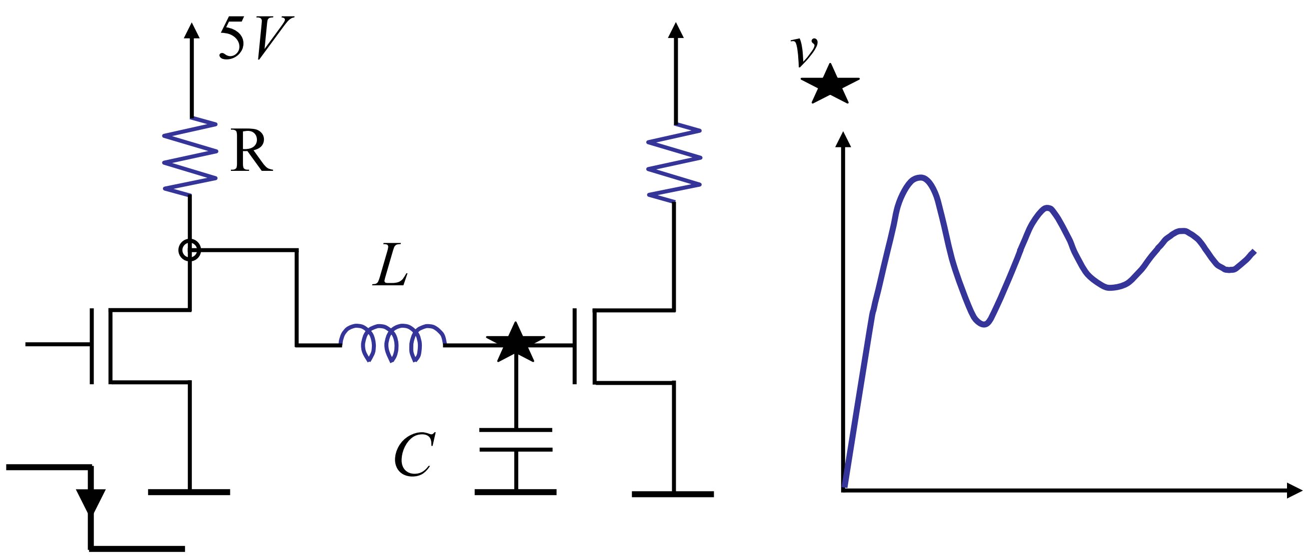
Sinusoidal Steady State

Review

- We now understand the why of:

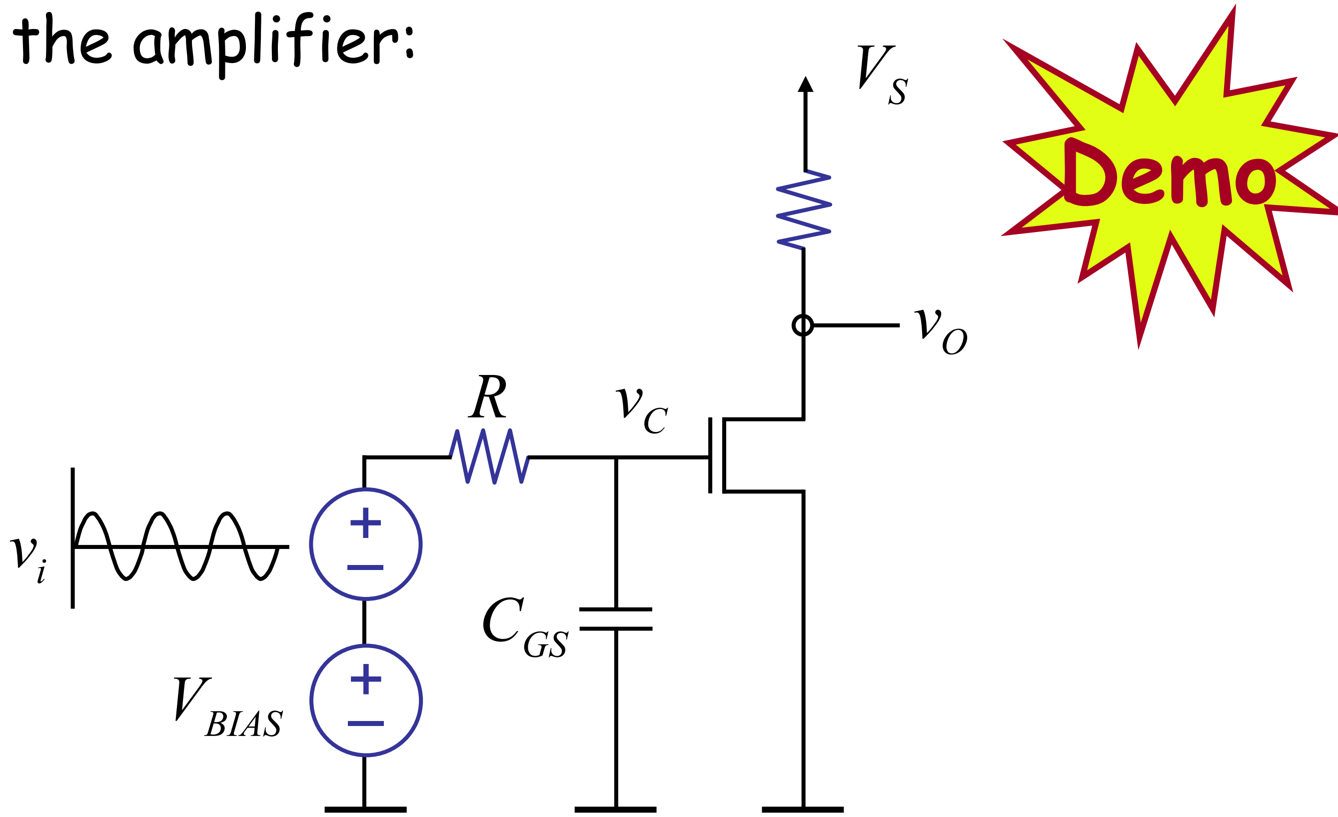


- Today, look at response of networks to sinusoidal drive.

Sinusoids important because signals can be represented as a sum of sinusoids. Response to sinusoids of various frequencies -- aka frequency response -- tells us a lot about the system

Motivation

For motivation, consider our old friend, the amplifier:



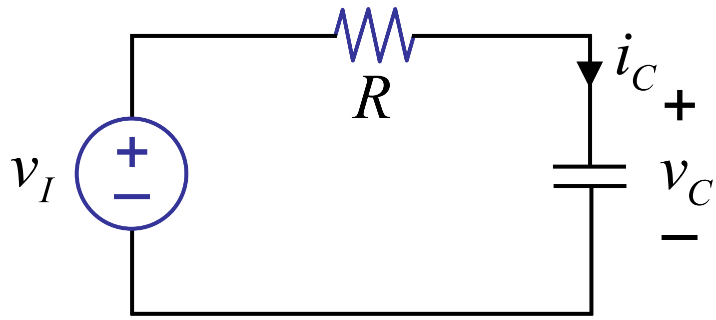
Observe v_o amplitude as the frequency of the input v_i changes. Notice it decreases with frequency.

Also observe v_o shift as frequency changes (phase).

Need to study behavior of networks for sinusoidal drive.

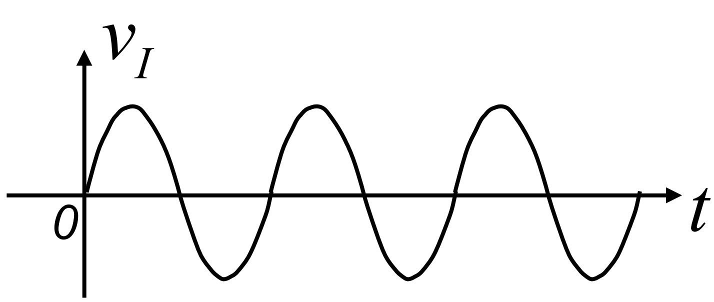
Sinusoidal Response of RC Network

Example:



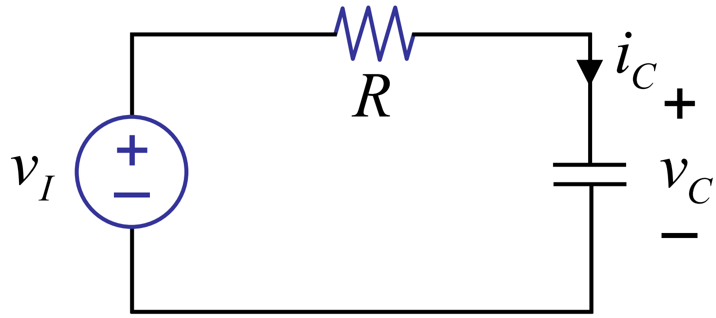
$$v_I(t) = V_i \cos \omega t \quad \text{for } t \geq 0 \quad (V_i \text{ real})$$
$$= 0 \quad \text{for } t < 0$$

$$v_C(0) = 0 \quad \text{for } t = 0$$

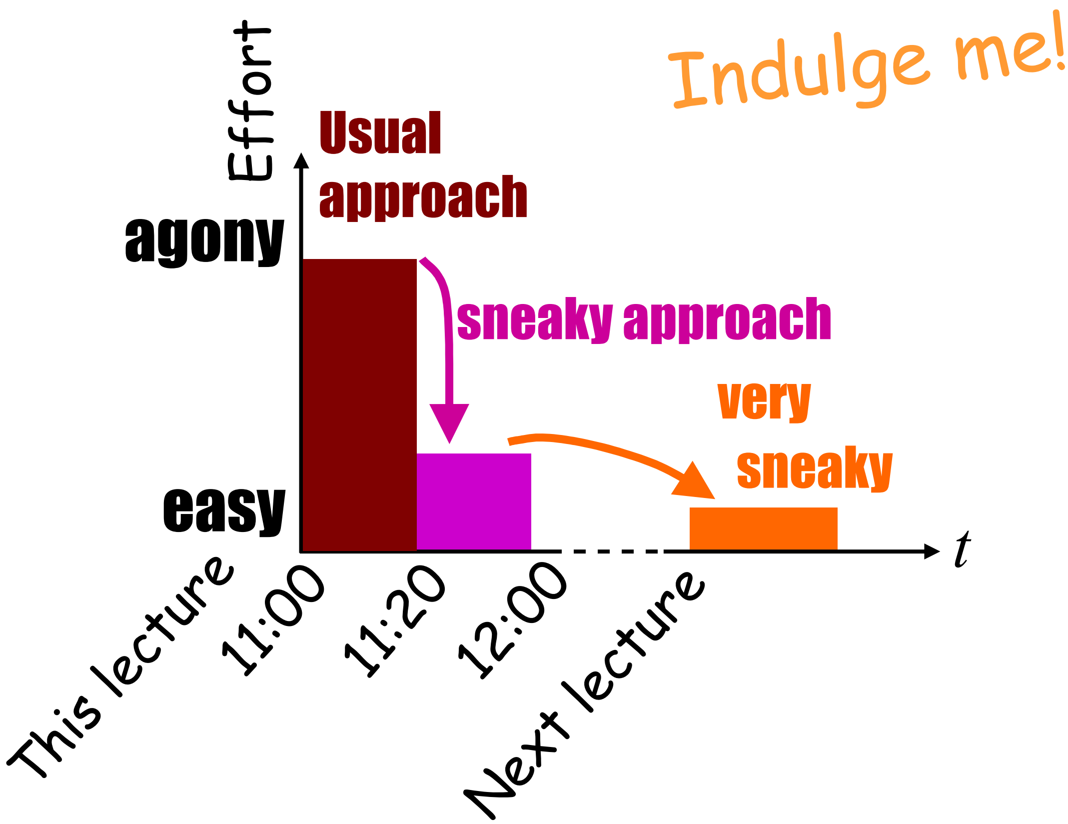


Our Approach

Example:



Determine $v_C(t)$



Let's use the usual approach...

1. Set up DE.
2. Find v_p.
3. Find v_H.
4. $v_C = v_P + v_H$, solve for unknowns using initial conditions

Usual approach...

1 Set up DE

$$RC\frac{dv_C}{dt}+v_C = v_I$$
$$= V_i \cos \omega t$$

That was easy!

② Find $\boxed{v_p}$

$$RC\frac{d\boxed{v_P}}{dt}+\boxed{v_P}=V_i\cos\omega t$$

First try: $\boxed{v_P}=A$ → nope

Second try: $\boxed{v_P}=A\cos\omega t$ → nope

Third try: $\boxed{v_P}=A\cos(\omega t+\phi)$ frequency

amplitude phase

$$-RCA\omega\sin(\omega t+\phi)+A\cos(\omega t+\phi)=V_i\cos\omega t$$

$$-RCA\omega\sin\omega t\cos\phi-RCA\omega\cos\omega t\sin\phi+$$
$$A\cos\omega t\cos\phi-A\sin\omega t\sin\phi\qquad=V_i\cos\omega t$$

⋮ **gasp !**

works, but trig nightmare!

Let's get sneaky!

Find particular solution to another input...

$$RC\frac{dv_{PS}}{dt} + v_{PS} = v_{IS}$$

$$= V_i e^{st}$$

(s: sneaky :-))

Try solution $v_{PS} = V_p e^{st}$

$$RC\frac{dV_p e^{st}}{dt} + V_p e^{st} = V_i e^{st}$$

$$sRCV_p \cancel{e^{st}} + V_p \cancel{e^{st}} = V_i \cancel{e^{st}}$$

$$(sRC + 1)V_p = V_i$$

$$\boxed{V_p = \frac{V_i}{1 + sRC}}$$

Nice property of exponentials

Thus, $v_{PS} = \frac{V_i}{1 + sRC} \cdot e^{st}$

is particular solution to $V_i e^{st}$

||||ly $\underbrace{\frac{V_i}{1 + j\omega RC}}_{V_p} \cdot e^{j\omega t} \longrightarrow$ solution for $V_i e^{j\omega t}$

where we replace $s = j\omega$

$V_p \longrightarrow$ complex amplitude

② Fourth try to find v_P...

using the sneaky approach

Fact 1: Finding the response to

$$V_i e^{j\omega t}$$

was easy.

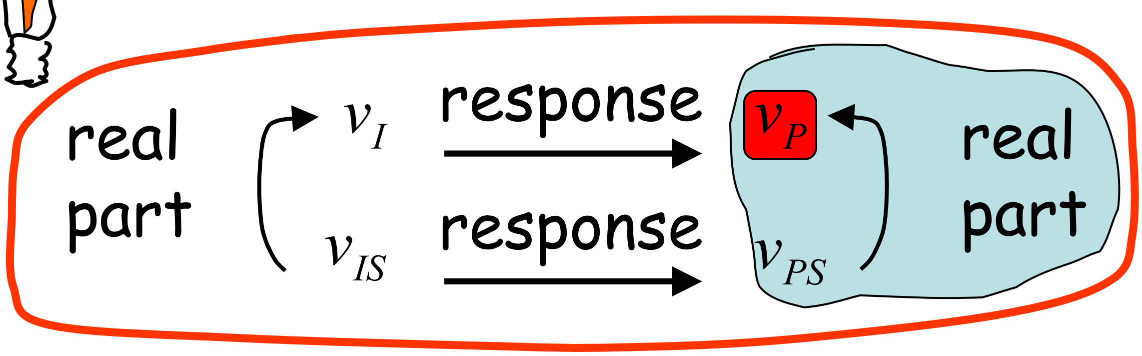
Fact 2:

$$v_I = V_i \cos \omega t$$

$$= \text{real}\left[V_i e^{j\omega t}\right] = \text{real}\left[v_{IS}\right]$$

from Euler relation,

$$e^{j\omega t} = \cos \omega t + j \sin \omega t$$



an inverse superposition argument, assuming system is real, linear.

② Fourth try to find $\boxed{v_P}$..

so,

$$\boxed{v_P} = Re\left[\underset{\text{complex}}{v_{PS}}\right] = Re\left[V_p e^{j\omega t}\right]$$

$$= \mathrm{Re}\left[\frac{V_i}{1 + j\omega RC} \cdot e^{j\omega t}\right]$$

$$= \mathrm{Re}\left[\frac{V_i(1 - j\omega RC)}{1 + \omega^2 R^2 C^2} \cdot e^{j\omega t}\right]$$

$$= Re\left[\frac{V_i}{\sqrt{1 + \omega^2 R^2 C^2}} \cdot e^{j\phi} e^{j\omega t}\right] \text{, } tan\,\phi = -\omega RC$$

$$= Re\left[\frac{V_i}{\sqrt{1 + \omega^2 R^2 C^2}} \cdot e^{j(\omega t + \phi)}\right]$$

$$\boxed{v_P} = \frac{V_i}{\sqrt{1 + \omega^2 R^2 C^2}} \cdot cos(\omega t + \phi)$$

Recall, $\boxed{v_P}$ is particular response to $V_i \cos \omega t$.

3 Find v_H

Recall, $v_H = Ae^{\frac{-t}{RC}}$

④ Find total solution

$$v_C = \boxed{v_P} + v_H$$

$$v_C = \frac{V_i}{\sqrt{1+\omega^2 R^2 C^2}} \cos(\omega t + \phi) + A e^{-\frac{t}{RC}}$$

$$\text{where} \quad \phi = \tan^{-1}(-\omega RC)$$

Given $v_C(0) = 0$ for $t = 0$

so,

$$A = -\frac{V_i}{\sqrt{1+\omega^2 R^2 C^2}} \cos(\phi)$$

Done! ***Phew*!**

Sinusoidal Steady State

We are usually interested only in the particular solution for sinusoids, i.e. after transients have died.

Notice when $t \to \infty, \quad v_C \to v_P$ as $e^{-\frac{t}{RC}} \to 0$

$$v_C = \frac{V_i}{\sqrt{1+\omega^2 R^2 C^2}} \cos(\omega t + \phi) + A e^{-\frac{t}{RC}}$$

0

$$\text{where} \quad \phi = \tan^{-1}(-\omega RC)$$

$$A = -\frac{V_i}{\sqrt{1+\omega^2 R^2 C^2}} \cos(\phi)$$

V_p $\qquad \angle V_p$

Described as

SSS: Sinusoidal Steady State

Sinusoidal Steady State

All information about SSS is contained in V_p, the complex amplitude!

Recall

$$V_p = \frac{V_i}{1 + j\omega RC}$$

Steps 3 , 4 were a waste of time!

$$\frac{V_p}{V_i} = \frac{1}{1 + j\omega RC}$$

$$\frac{V_p}{V_i} = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}} e^{j\phi} \text{ where}$$

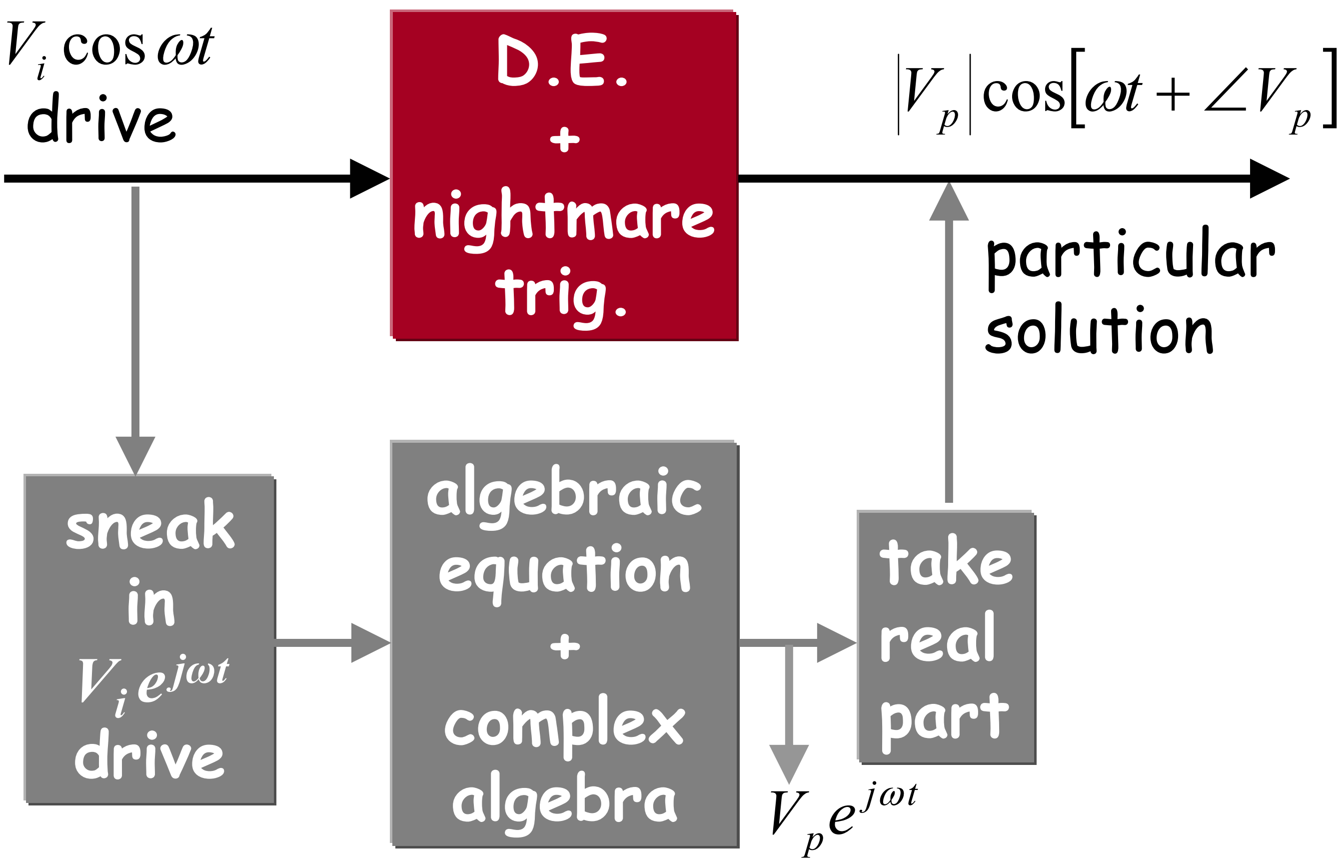
$$\phi = \tan^{-1} - \omega RC$$

magnitude $\left| \frac{V_p}{V_i} \right| = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}}$

phase ϕ: $\angle \frac{V_p}{V_i} = -\tan^{-1} \omega RC$

Sinusoidal Steady State

Visualizing the process of finding the particular solution v_P



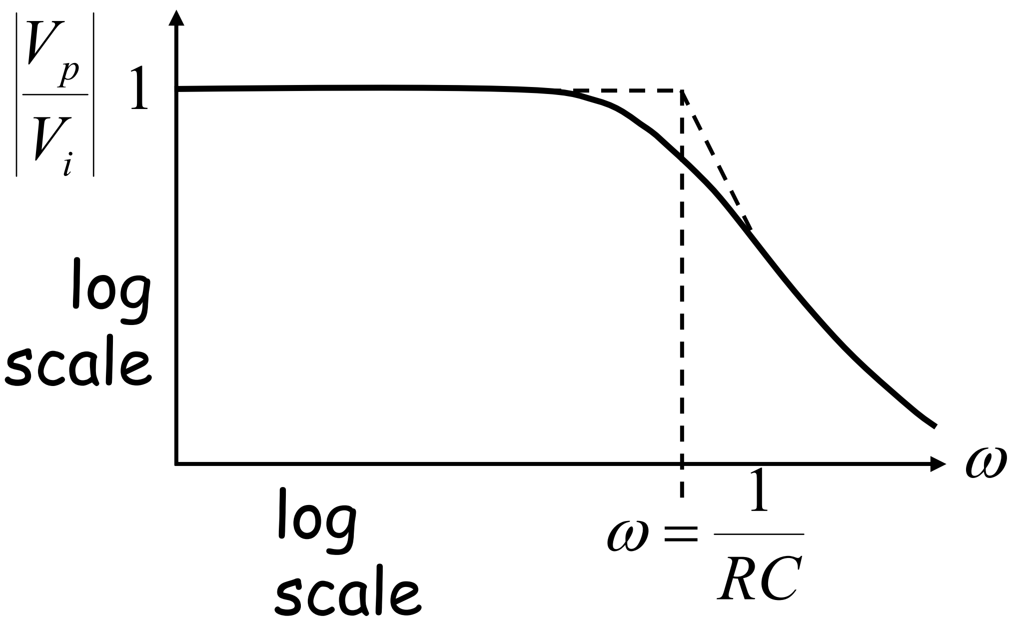
the sneaky path!

Magnitude Plot

transfer function

$$H(j\omega) = \frac{V_p}{V_i}$$

$$\left|\frac{V_p}{V_i}\right| = \frac{1}{\sqrt{1+\omega^2 R^2 C^2}}$$



From demo: explains v_o fall off for high frequencies!

Phase Plot

$$\phi = tan^{-1} - \omega RC$$