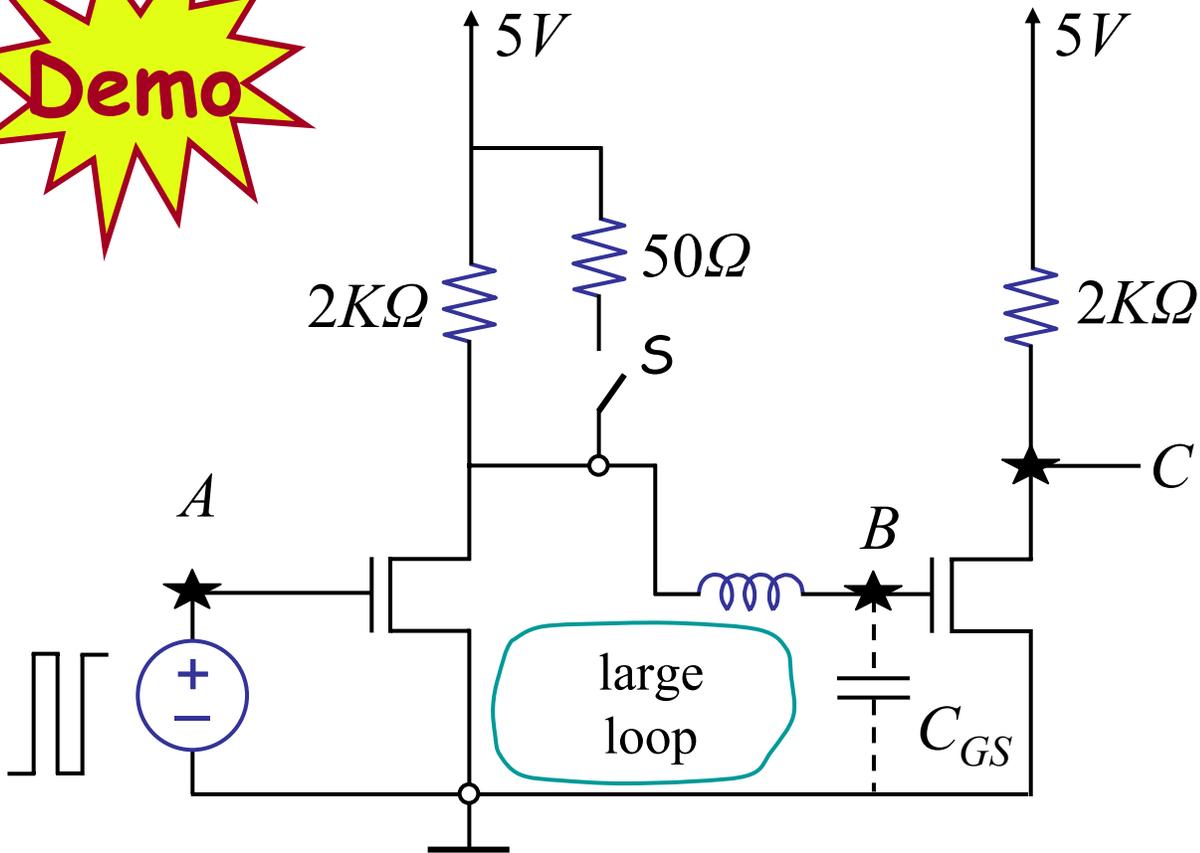


6.002

**CIRCUITS AND
ELECTRONICS**

Second-Order Systems

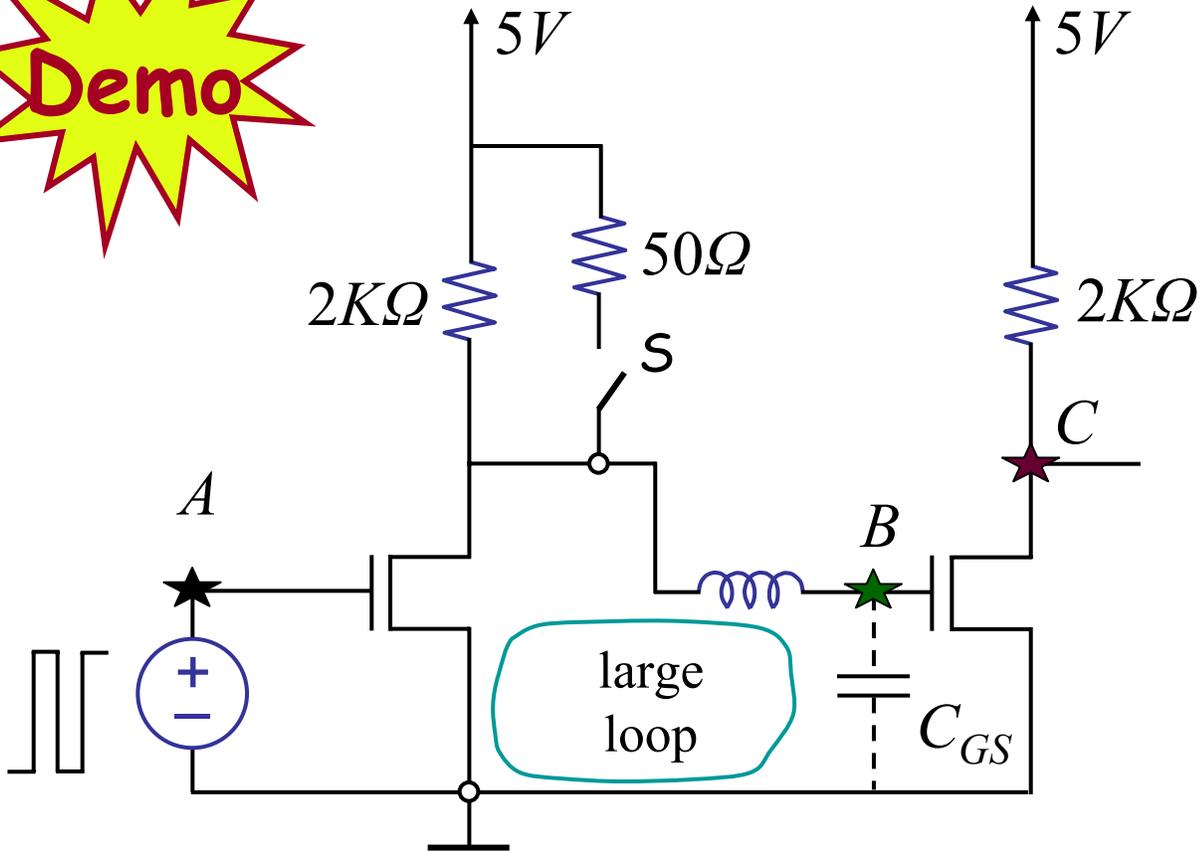
Second-Order Systems



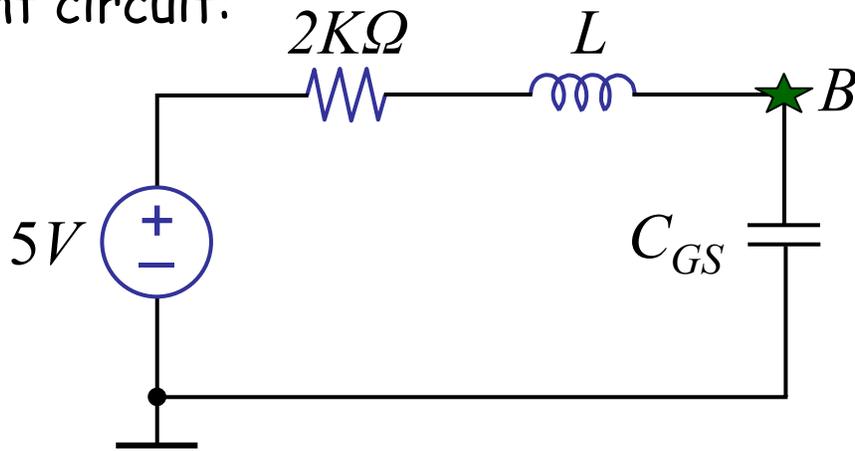
Our old friend, the inverter, driving another. The parasitic inductance of the wire and the gate-to-source capacitance of the MOSFET are shown

[Review complex algebra appendix for next class]

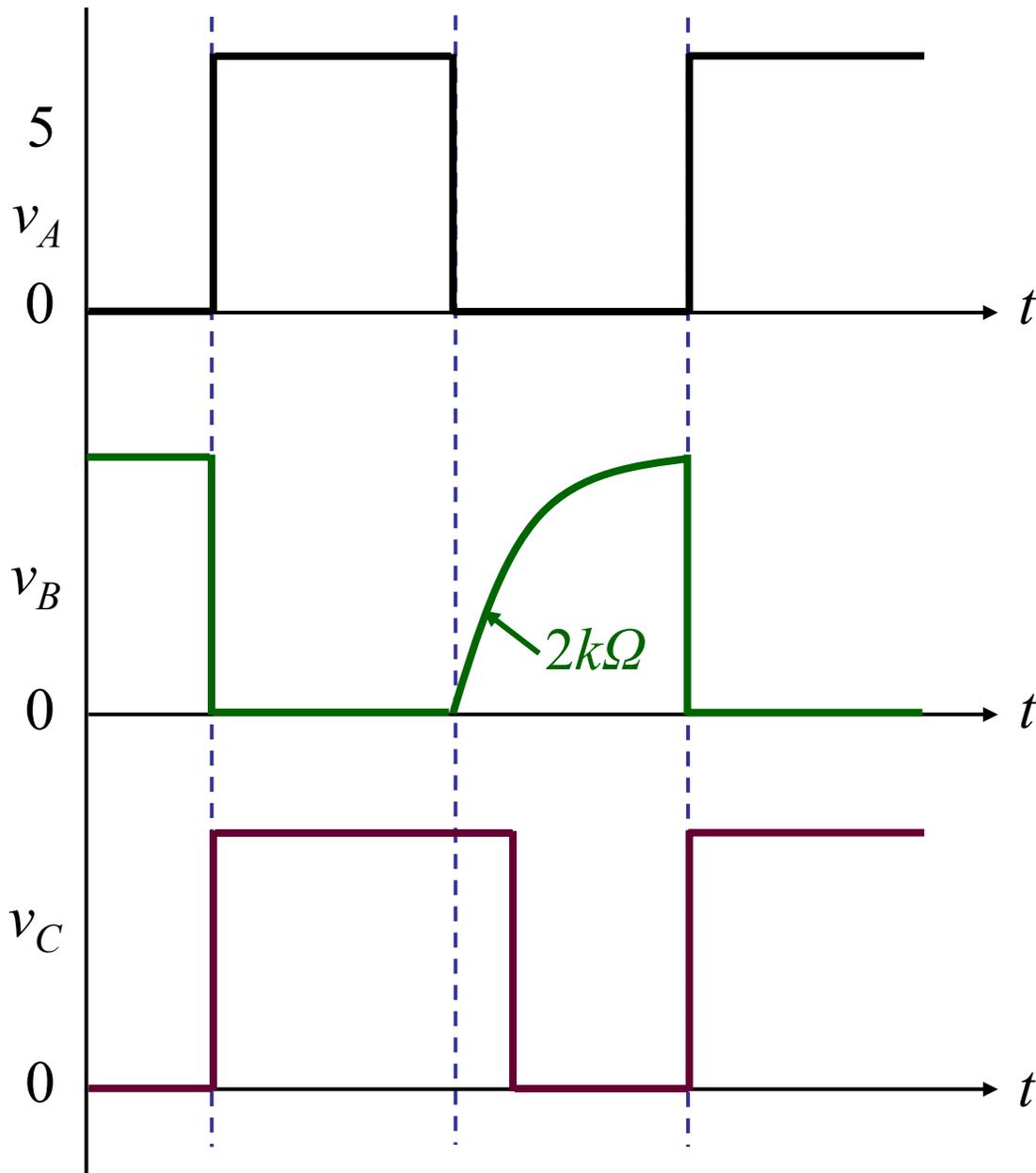
Second-Order Systems



Relevant circuit:

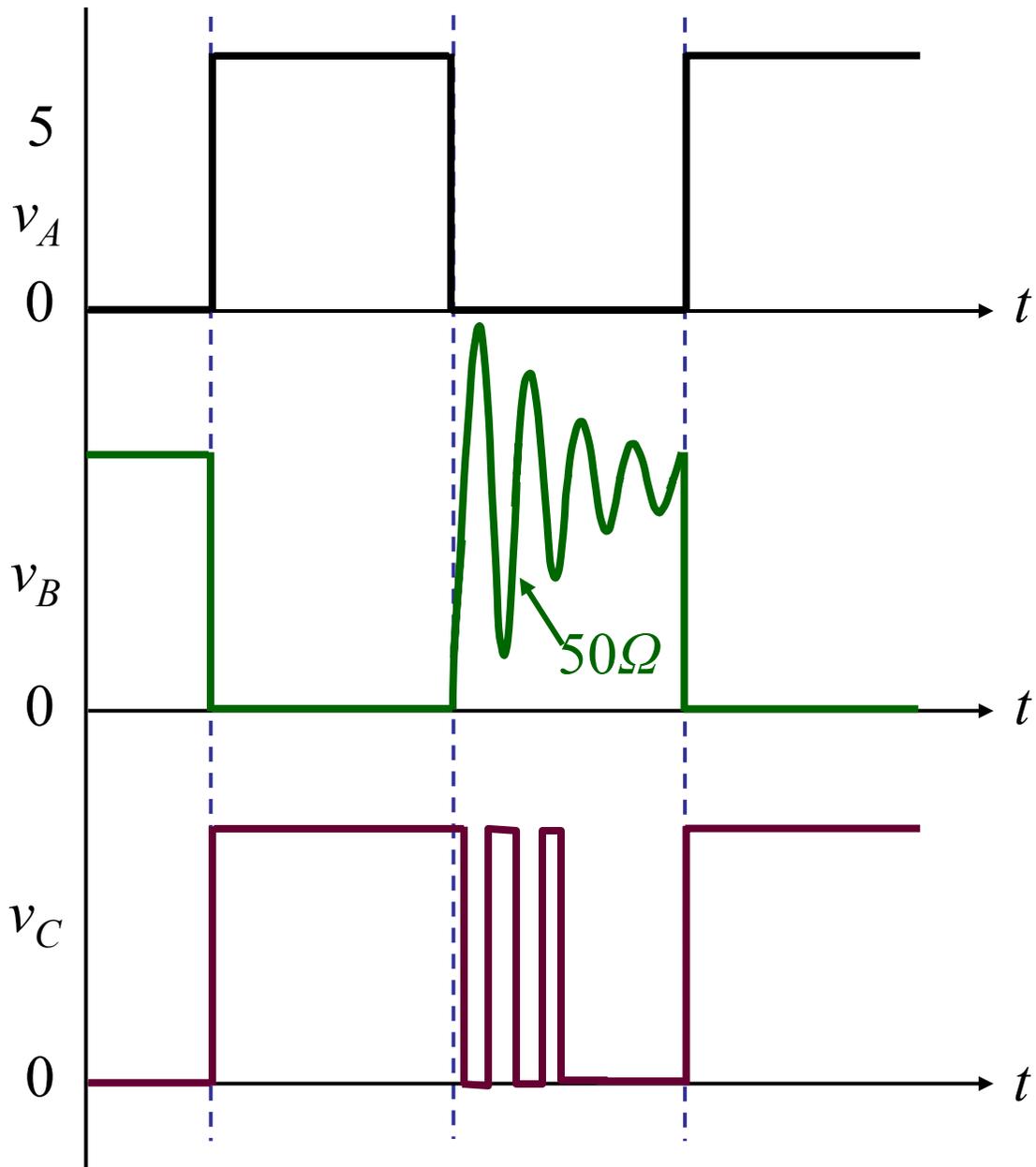


Observed Output $2k\Omega$



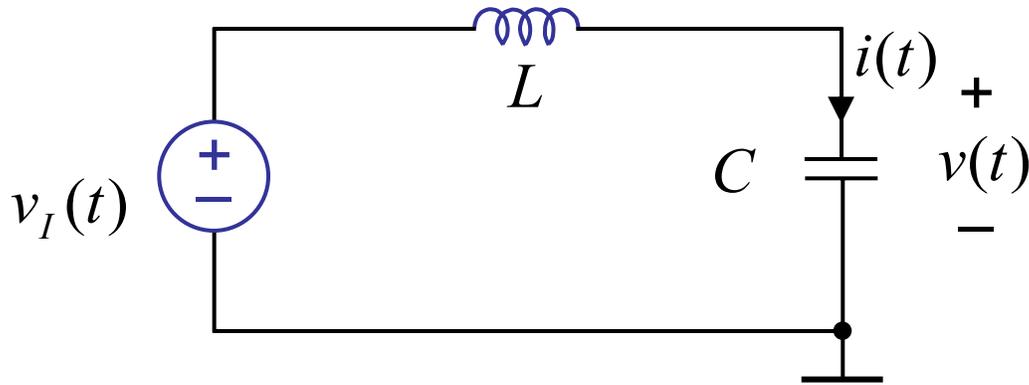
Now, let's try to speed up our inverter by closing the switch S to lower the effective resistance

Observed Output $\sim 50\Omega$



Huh!

First, let's analyze the LC network



Node method:

$$i(t) = C \frac{dv}{dt}$$

$$\frac{1}{L} \int_{-\infty}^t (v_I - v) dt = C \frac{dv}{dt}$$

$$\frac{1}{L} (v_I - v) = C \frac{d^2v}{dt^2}$$

$$LC \frac{d^2v}{dt^2} + v = v_I$$

$\underbrace{\hspace{2cm}}$
time²

v, i state variables

Recall

$$v_I - v = L \frac{di}{dt}$$

$$\frac{1}{L} \int_{-\infty}^t (v_I - v) dt = i$$

Solving

Recall, the method of homogeneous and particular solutions:

- ① Find the particular solution.
- ② Find the homogeneous solution.

↓
4 steps

- ③ The total solution is the sum of the particular and homogeneous.

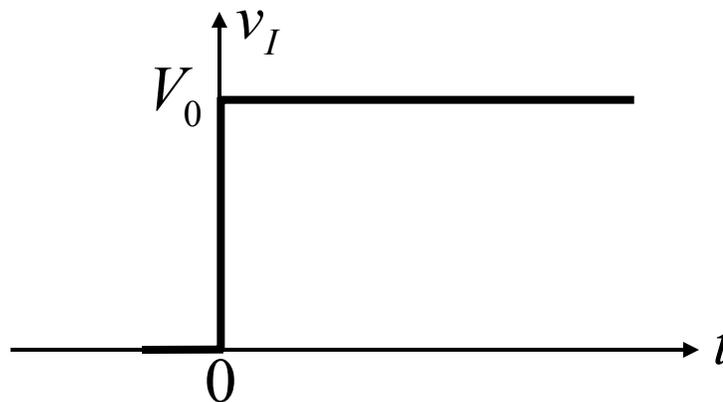
Use initial conditions to solve for the remaining constants.

$$v = v_P(t) + v_H(t)$$

Let's solve

$$LC \frac{d^2 v}{dt^2} + v = v_I$$

For input



And for initial conditions

$$v(0) = 0 \quad i(0) = 0 \quad [\text{ZSR}]$$

① Particular solution

$$LC \frac{d^2 v_P}{dt^2} + v_P = V_0$$

$v_P = V_0$ is a solution.

② Homogeneous solution

Solution to

$$LC \frac{d^2 v_H}{dt^2} + v_H = 0$$

Recall, v_H : solution to homogeneous equation (drive set to zero)

Four-step method:

① Assume solution of the form*

$$v_H = Ae^{st}, \quad A, s = ?$$

$$\text{so, } LC \cancel{A} s^2 \cancel{e^{st}} + \cancel{A} e^{st} = 0$$

② $s^2 = -\frac{1}{LC}$ { characteristic equation

$$s = \pm j \sqrt{\frac{1}{LC}}$$

$$j = \sqrt{-1}$$

③ Roots

$$s = \pm j\omega_o$$

$$\omega_o = \sqrt{\frac{1}{LC}}$$

General solution,

④ $v_H = A_1 e^{j\omega_o t} + A_2 e^{-j\omega_o t}$

*Differential equations are commonly solved by guessing solutions

③ Total solution

$$v(t) = v_P(t) + v_H(t)$$

$$v(t) = V_0 + A_1 e^{j\omega_0 t} + A_2 e^{-j\omega_0 t}$$

Find unknowns from initial conditions.

$$v(0) = 0$$

$$0 = V_0 + A_1 + A_2$$

$$i(0) = 0$$

$$i(t) = C \frac{dv}{dt}$$

$$i(t) = CA_1 j\omega_0 e^{j\omega_0 t} - CA_2 j\omega_0 e^{-j\omega_0 t}$$

$$\text{so, } 0 = CA_1 j\omega_0 - CA_2 j\omega_0$$

$$\text{or, } A_1 = A_2$$

$$-V_0 = 2A$$

$$A_1 = -\frac{V_0}{2}$$

$$\text{so, } v(t) = V_0 - \frac{V_0}{2} (e^{j\omega_0 t} + e^{-j\omega_0 t})$$

③ Total solution

Remember Euler relation

$$e^{jx} = \cos x + j \sin x$$

(verify using Taylor's expansion)

$$\frac{e^{jx} + e^{-jx}}{2} = \cos x$$

so,

$$v(t) = V_0 - V_0 \cos \omega_o t$$

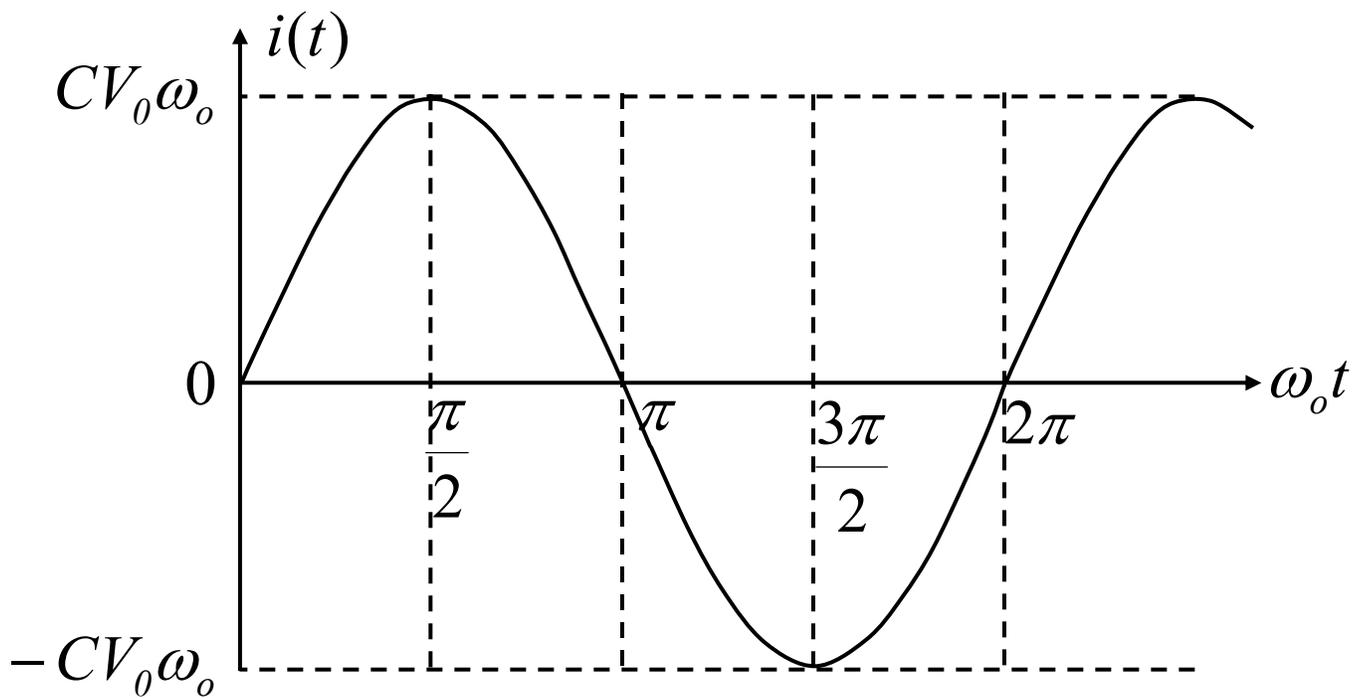
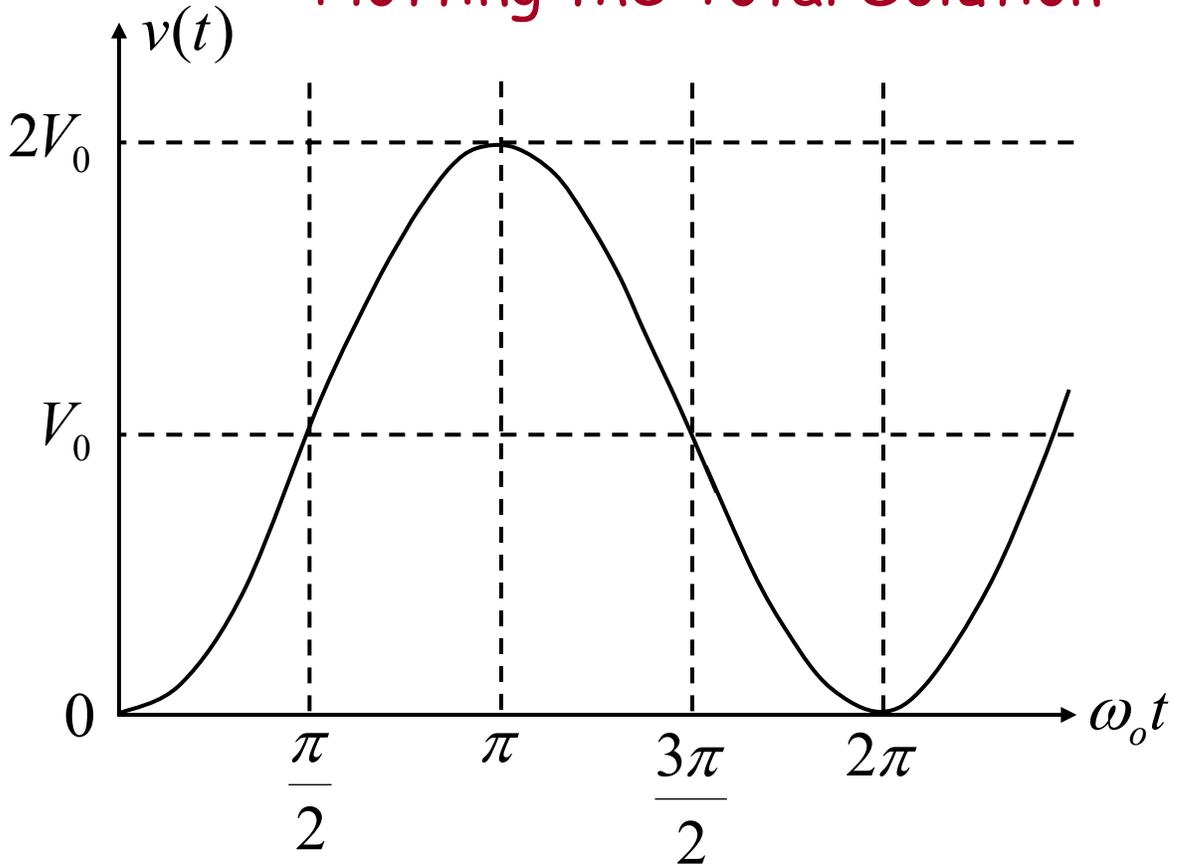
$$i(t) = CV_0 \omega_o \sin \omega_o t$$

where

$$\omega_o = \frac{1}{\sqrt{LC}}$$

The output looks sinusoidal

Plotting the Total Solution

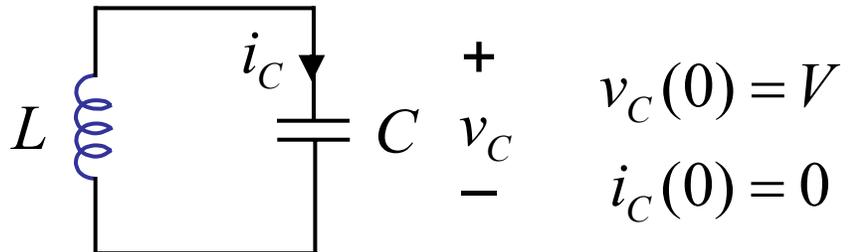


Summary of Method

- ① Write DE for circuit by applying node method.
- ② Find particular solution v_P by guessing and trial & error.
- ③ Find homogeneous solution v_H
 - Ⓐ Assume solution of the form Ae^{st} .
 - Ⓑ Obtain characteristic equation.
 - Ⓒ Solve characteristic equation for roots s_i .
 - Ⓓ Form v_H by summing $A_i e^{s_i t}$ terms.
- ④ Total solution is $v_P + v_H$, solve for remaining constants using initial conditions.

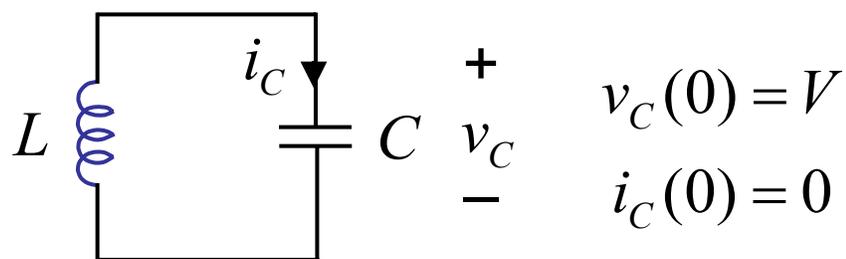
Example

What if we have:



We can obtain the answer directly from the homogeneous solution ($V_0 = 0$).

Example



We can obtain the answer directly from the homogeneous solution ($V_0 = 0$).

$$v_C(t) = A_1 e^{j\omega_o t} + A_2 e^{-j\omega_o t}$$

$$v_C(0) = V$$

$$V = A_1 + A_2$$

$$i_C(0) = 0$$

$$0 = CA_1 j\omega_o - CA_2 j\omega_o$$

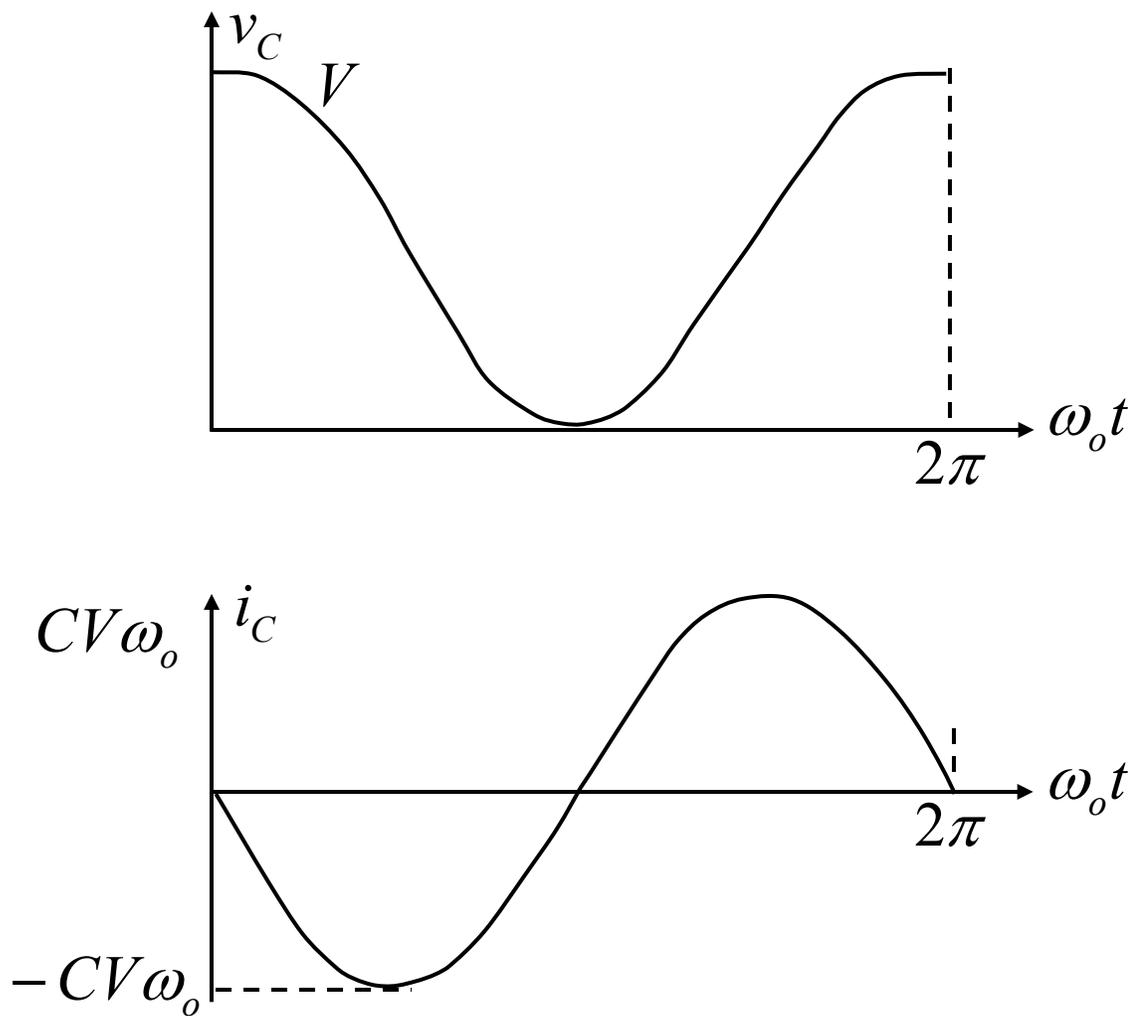
$$\text{or } A_1 = A_2 = \frac{V}{2}$$

$$\text{or } v_C = \frac{V}{2} (e^{j\omega_o t} + e^{-j\omega_o t})$$

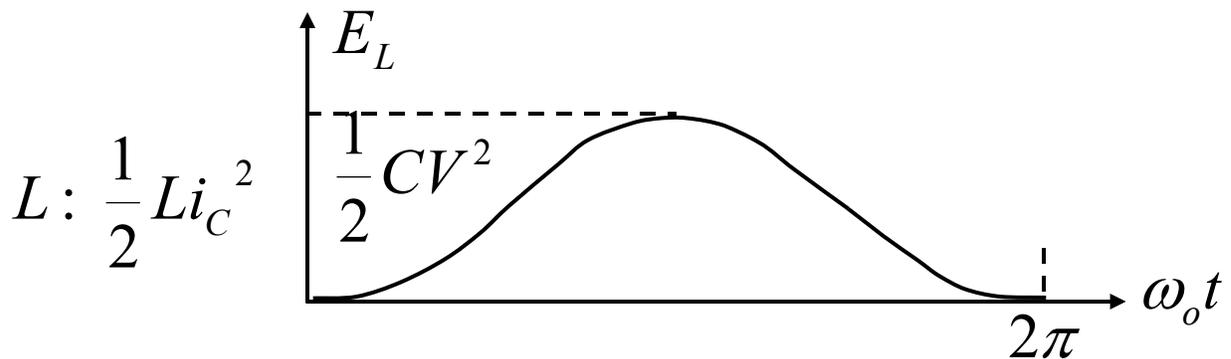
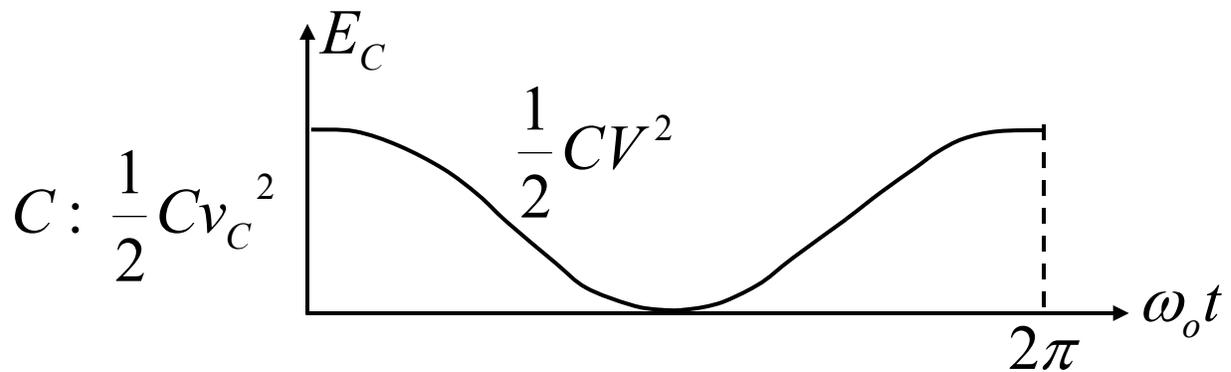
$$v_C = V \cos \omega_o t$$

$$i_C = -CV \omega_o \sin \omega_o t$$

Example



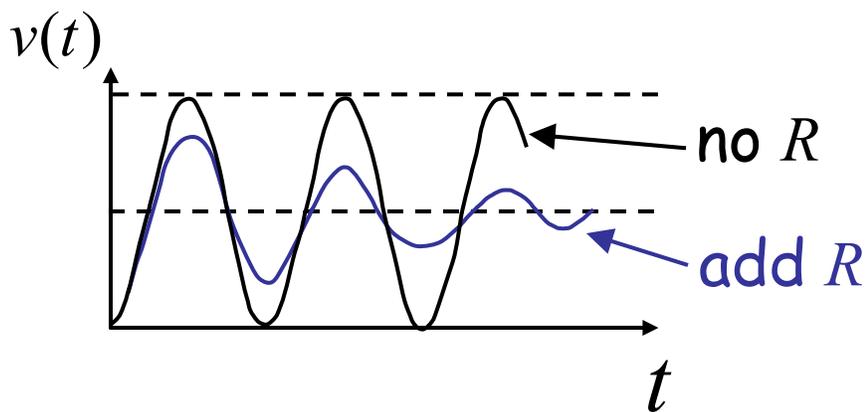
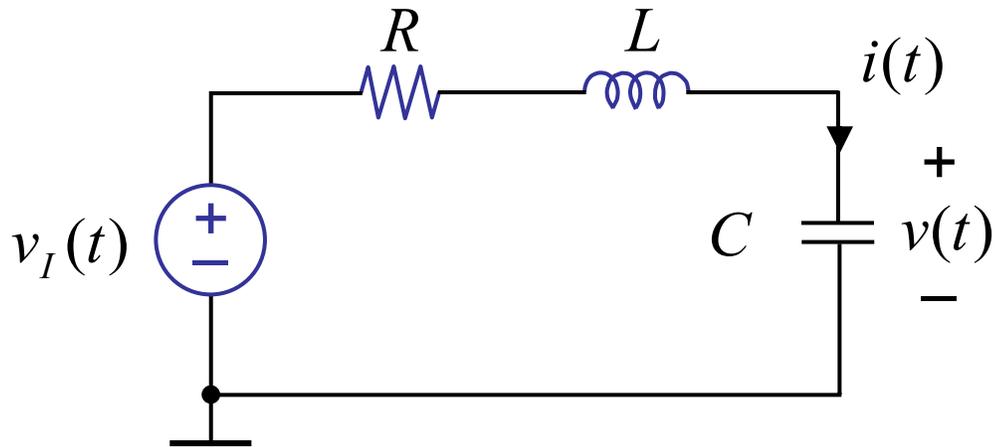
Energy



Notice $\frac{1}{2} C v_C^2 + \frac{1}{2} L i_C^2 = \frac{1}{2} C V^2$

Total energy in the system is a constant,
but it sloshes back and forth between the
Capacitor and the inductor

RLC Circuits



Damped sinusoids with R - remember demo!

See A&L Section 12.2